

# Size Ramsey Numbers of Stars versus Cliques

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## Abstract

The size Ramsey number  $\hat{r}(G, H)$  of two graphs  $G$  and  $H$  is the smallest integer  $m$  such that there exists a graph  $F$  on  $m$  edges with the property that every red-blue colouring of the edges of  $F$ , yields a red copy of  $G$  or a blue copy of  $H$ . In 1981, Erdős observed that  $\hat{r}(K_{1,k}, K_3) \leq \binom{2k+1}{2} - \binom{k}{2}$  and he conjectured that the corresponding upper bound on  $\hat{r}(K_{1,k}, K_3)$  is sharp. In 1983, Faudree and Sheehan extended this conjecture as follows:

$$\hat{r}(K_{1,k}, K_n) = \begin{cases} \binom{k(n-1)+1}{2} - \binom{k}{2} & k \geq n \text{ or } k \text{ odd.} \\ \binom{k(n-1)+1}{2} - k(n-1)/2 & \text{otherwise.} \end{cases}$$

They proved the case  $k = 2$ . In 2001, Pikhurko showed that this conjecture is not true for  $n = 3$  and  $k \geq 5$ , disproving the mentioned conjecture of Erdős. Here we prove Faudree and Sheehan's conjecture for a given  $k \geq 2$  and  $n \geq k^3 + 2k^2 + 2k$ .

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## 1 Introduction

Given two graphs  $G$  and  $H$ , we say that  $F \rightarrow (G, H)$ , if for any red-blue colouring of the edges of  $F$  we have a red copy of  $G$  or a blue copy of  $H$ . The *size Ramsey number*  $\hat{r}(G, H)$  of two graphs  $G$  and  $H$  is the minimum number of edges of a graph  $F$  such that  $F \rightarrow (G, H)$ . Using this notation, the *Ramsey number*  $r(G, H)$  is the minimum integer  $n$  such that  $K_n \rightarrow (G, H)$ . We also define the *restricted size Ramsey number*  $\hat{r}^*(G, H)$  for two graphs  $G$  and  $H$  as follows:

$$\hat{r}^*(G, H) = \min\{|E(F)| : F \rightarrow (G, H), |V(F)| = r(G, H)\}.$$

Clearly for every two graphs  $G$  and  $H$ , we have  $\hat{r}(G, H) \leq \hat{r}^*(G, H)$ . Also, by the definition of  $r(G, H)$ , we have  $K_{r(G, H)} \rightarrow (G, H)$ . Since the complete graph on  $r(G, H)$  vertices

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has  $\binom{r(G,H)}{2}$  edges, we obtain trivially

$$\hat{r}(G, H) \leq \binom{r(G, H)}{2}. \quad (1.1)$$

Chvátal showed that equality holds in (1.1), when  $G$  and  $H$  are complete graphs (see [2]).

The investigation of the size Ramsey numbers of graphs was initiated by Erdős et al. [2] in 1978. In this paper, we investigate the size Ramsey number of  $K_{1,k}$ , the star with  $k$  edges, versus the complete graph  $K_n$ . These numbers were first considered by Erdős et al. [2]. They showed the following asymptotic result:

**Theorem 1.1.** [2] *Let  $\varepsilon$  be a fixed real number satisfying  $0 < \varepsilon < 1$  and let  $n \geq 3$  be a fixed natural number. If  $k$  is sufficiently large, then*

$$\hat{r}(K_{1,k}, K_n) \geq \max\{k^2/2, (1 - \varepsilon)\lfloor(n - 2)^2/4\rfloor k^2/2\}.$$

Let the graph  $K_{k+1} + \overline{K}_k$  be obtained from  $K_{k+1}$  by considering  $k$  new vertices and joining each vertex of  $K_{k+1}$  to all these  $k$  additional vertices. In [1], Erdős observed that  $K_{k+1} + \overline{K}_k \rightarrow (K_{1,k}, K_3)$  and conjectured that the corresponding upper bound on  $\hat{r}(K_{1,k}, K_3)$  is sharp. Faudree and Sheehan generalized this result and showed that:

**Theorem 1.2.** [4] *Let  $k, n \geq 2$ , then*

$$\hat{r}^*(K_{1,k}, K_n) = \begin{cases} \binom{k(n-1)+1}{2} - \binom{k}{2} & k \geq n \text{ or } k \text{ odd,} \\ \binom{k(n-1)+1}{2} - k(n-1)/2 & \text{otherwise.} \end{cases}$$

They also posed the following conjecture, generalizing the mentioned conjecture of Erdős on  $\hat{r}(K_{1,k}, K_3)$ .

**Conjecture 1.3.** [4] *Let  $k, n \geq 2$ . Then  $\hat{r}(K_{1,k}, K_n) = \hat{r}^*(K_{1,k}, K_n)$ .*

They proved the case  $k = 2$  of this conjecture (see [4]). Pikhurko [5], with a nice counterexample, disproved the Erdős conjecture on  $\hat{r}(K_{1,k}, K_3)$  for  $k \geq 5$  (the case  $n = 3$  of Conjecture 1.3). More precisely, he showed that  $\hat{r}(K_{1,k}, K_3) < k^2 + \sqrt{2}k^{3/2} + k$  for  $k \geq 1$ . One can easily check that for  $k \geq 5$ , we have

$$k^2 + \sqrt{2}k^{3/2} + k < \binom{2k+1}{2} - \binom{k}{2}.$$

Also, Pikhurko [6] showed that for any graph  $F$  with chromatic number  $\chi(F) \geq 4$ ,

$$\hat{r}(K_{1,k}, F) \leq \chi(F)(\chi(F) - 2)k^2/2 + o(k^2)$$

and he conjectured that this is sharp. He proved his conjecture for the case  $\chi(F) = 4$ .

In this paper, we show that for a fixed  $k \geq 2$ , Conjecture 1.3 holds for  $n \geq k^3 + 2k^2 + 2k$ . More precisely, we demonstrate the following theorem.

**Theorem 1.4.** *Let  $k \geq 2$  and  $n \geq k^3 + 2k^2 + 2k$ . Then*

$$\hat{r}(K_{1,k}, K_n) = \hat{r}^*(K_{1,k}, K_n) = \begin{cases} \binom{k(n-1)+1}{2} - \binom{k}{2} & \text{if } k \text{ is odd,} \\ \binom{k(n-1)+1}{2} - k(n-1)/2 & \text{if } k \text{ is even.} \end{cases}$$

Note that, we also make no attempt to give out a better lower bound for  $n$  in terms on  $k$  in Theorem 1.4. Throughout the paper, for the sake of clarity of presentation, we omit floor and ceiling signs whenever they are not crucial.

**Conventions and Notations:** For a graph  $G$ , we write  $V(G)$ ,  $E(G)$  and  $e(G)$  for the vertex set, edge set and the number of edges of  $G$ , respectively. For  $v \in V(G)$ , by  $N_G(v)$  we mean the set of all neighbors of  $v$  and the degree  $d_G(v)$  of  $v$  is  $d_G(v) = |N_G(v)|$ . We denote by  $\delta(G)$  and  $\Delta(G)$  the minimum and maximum degrees of  $G$ , respectively. Let  $X \subseteq V(G)$ . Then  $G[X]$  is the induced subgraph of  $G$  with vertex set  $X$ . We write  $G \setminus X$  for  $G[V(G) \setminus X]$ . Let  $A, B \subset V(G)$ , then  $e(A, B) = |\{\{x, y\} \in E(G) : x \in A, y \in B\}|$ , is the number of edges connecting a vertex of  $A$  to a vertex of  $B$ . By  $\overline{G}$  we mean the complement of  $G$ .

## 2 Preliminaries

In this section, we prove some results that will be used in the follow up section. We also recall some results from [4] and [6]. The following theorem is indeed a special case of Theorem 3.1 of [6].

**Theorem 2.1.** [6] *Let  $k \geq 2$  and  $n \geq 1$ . If  $G$  is a graph so that  $G \longrightarrow (K_{1,k}, K_n)$ , then  $e(G) \geq k^2 \binom{n-1}{2}$ .*

we also use the following lemma of Pikhurko [6, Lemma 5.1].

**Lemma 2.2.** [6] *Let  $G$  be a graph so that  $G \longrightarrow (K_{1,k}, K_n)$ . For any set  $S \subset V(G)$ , there exists  $T \subset V(G)$  such that  $\Delta(G \setminus T) < k$ , each vertex of  $T$  sends at least  $k$  edges to  $V(G) \setminus T$  and  $T$  is incident to at least  $k(|T| - |S|) + e(S, V(G))$  edges.*

The following lemma is a modified version of [4, Lemma 1]. But, for the sake of completeness, we state a proof here.

**Lemma 2.3.** [4] *Let  $k \geq 2$  and  $G$  be a graph with  $e(G) \geq \binom{k}{2} + 1$ . Then either*

- (i)  $G$  contains an induced subgraph with  $k+1$  vertices and minimum degree at least 1 or
- (ii)  $G$  contains a matching  $M$  with  $|M| = e(G)$ .

*Proof.* We use induction on  $k$ . The case  $k = 2$  is easily verified. Suppose that  $k \geq 3$ . If there is a vertex  $v \in V(G)$  so that  $d_G(v) \geq k$ , then the induced subgraph on  $A \cup \{v\}$  has  $k+1$  vertices with minimum degree at least one, where  $A \subset N_G(v)$  is a set containing  $k$  vertices. Hence we may assume that  $d_G(v) \leq k-1$ , for all  $v \in V(G)$ . Furthermore, there exists  $v \in V(G)$  such that  $d_G(v) \geq 2$ , otherwise Lemma 2.3 (ii) holds. Now, choose  $v \in V(G)$  so that  $2 \leq d_G(v) \leq k-1$ . Set  $G' = G \setminus \{v\}$ . Clearly  $e(G') \geq e(G) - (k-1) \geq \binom{k-1}{2} + 1$ .

So by the induction hypothesis  $V(G')$  contains a subset  $Y$  so that, either

(a)  $|Y| = k$  and  $\delta(G'[Y]) \geq 1$ , or

(b)  $G'[Y] \cong sK_2$ , so that  $s = e(G')$ .

At first assume that  $N_G(v) \cap Y \neq \emptyset$ . If  $Y$  is of type (a), then set  $X = Y \cup \{v\}$  and Lemma 2.3 (i) holds. So assume that  $Y$  is of type (b). If  $k$  is odd, then the set of  $k + 1$  vertices incident to some set of  $(k + 1)/2$  disjoint edges contained in  $G'[Y]$  satisfies Lemma 2.3 (i). Now suppose that  $k$  is even and  $u \in N_G(v) \cap Y$ . Let  $X$  be the set of  $k$  vertices incident to  $k/2$  disjoint edges in  $G'[Y]$  including an edge incident to  $u$ . Clearly  $X \cup \{v\}$  satisfies Lemma 2.3 (i).

Now assume that  $N_G(v) \cap Y = \emptyset$ . Let  $v_1, v_2 \in N_G(v)$ . First, suppose that  $Y$  is of type (b). If  $k$  is odd, then by an argument similar to the previous paragraph we can find a subgraph in  $G$  with  $k + 1$  vertices and minimum degree at least 1. If  $k$  is even, then let  $X^*$  be the set of vertices incident to some subset of  $(k - 2)/2$  disjoint edges in  $G'[Y]$ . Set  $U = X^* \cup \{v, v_1, v_2\}$ . Clearly  $G[U]$  satisfies Lemma 2.3 (i). So we may assume that  $Y$  is of type (a). Choose any vertex  $u' \in Y$  and write  $X = (Y \setminus \{u'\}) \cup \{v, v_1\}$ . If  $X$  does not satisfy Lemma 2.3 (i), then there exists  $x \in X$  so that  $d_{G[X]}(x) = 0$ . Since  $Y$  is of type (a), so  $x \in Y \setminus \{u'\}$  and  $x \sim u'$  in  $G$ . So we have proved that any vertex  $u' \in Y$  is incident to some vertex  $x \in Y$  so that  $d_{G'[Y]}(x) = 1$ . In particular this is true for each vertex  $x$  with  $d_{G'[Y]}(x) = 1$ . Hence,  $G'[Y]$  is a matching and  $k$  is even. Now set  $U = X^* \cup \{v, v_1, v_2\}$  where  $X^*$  is the set of  $k - 2$  vertices incident to  $(k - 2)/2$  disjoint edges in  $G'[Y]$ . Clearly  $G[U]$  satisfies Lemma 2.3 (i). So we are done.  $\square$

The following result is an immediate consequence of Lemma 2.3.

**Corollary 2.4.** [4, Lemma 2] *Let  $k \geq 3$  and  $G$  be a graph with  $e(G) \geq \binom{k}{2} + 1$ . If  $k$  is odd, then  $G$  contains an induced subgraph with  $k + 1$  vertices and minimum degree at least 1.*

**Remark 2.5.** *Let  $G$  be a graph so that  $G \rightarrow (K_{1,k}, K_n)$ . If  $G$  is edge minimal, then each vertex of  $G$  must be in some clique  $K_n$ . Otherwise, assume that some vertex  $v \in V(G)$  is not in any clique  $K_n$ . Colour the edges of  $G' = G \setminus \{v\}$  red or blue arbitrarily and extend this colouring to  $G$  by colouring the edges incident to  $v$  blue. Since  $v$  is not in any blue copy of  $K_n$ , so  $G'$  has either a red copy of  $K_{1,k}$  or a blue copy of  $K_n$  and so  $G' \rightarrow (K_{1,k}, K_n)$ , which is a contradiction with the edge minimality of  $G$ .*

**Lemma 2.6.** *Let  $k \geq 2$  and  $n \geq 3k + 3$ . Let  $H$  be a graph with  $R + t$  vertices, where  $R = r(K_{1,k}, K_n) = k(n - 1) + 1$  and  $0 \leq t \leq \lfloor \frac{kn - 2k^2}{(k + 1)^2} \rfloor$ . Set*

$$R' = \begin{cases} \binom{k}{2} + 1 & \text{if } k \text{ is odd,} \\ k(n - 1)/2 + 1 & \text{if } k \text{ is even.} \end{cases}$$

*If  $e(H) \geq Rt + \binom{t}{2} + R'$ , then  $H$  contains  $t + 1$  disjoint subsets  $A_1, \dots, A_{t+1}$  of vertices so that for  $1 \leq i \leq t + 1$ , we have  $|A_i| = k + 1$  and  $\delta(H[A_i]) \geq 1$ .*

*Proof.* We use induction on  $t$ . First let  $t = 0$ . If there is no subset  $A_1 \subseteq V(H)$  with  $|A_1| = k + 1$  and  $\delta(H[A_1]) \geq 1$ , then using Lemma 2.3 and Corollary 2.4 we may assume

that  $k$  is even and  $H$  contains a matching  $M$  with  $|M| = e(H)$ . But it is impossible, since  $|V(H)| = k(n-1) + 1$  and  $e(H) \geq k(n-1)/2 + 1$ . Now, let  $t \geq 1$ . Set

$$A = \{v \in V(H) : d_H(v) \geq (k+1)(t+1)\}.$$

We have two following cases:

**Case 1.**  $A \neq \emptyset$ .

Set  $H' = H \setminus \{v\}$ , where  $v \in A$ . Since  $d_H(v) \leq R + t - 1$ , we have

$$e(H') \geq R(t-1) + \binom{t-1}{2} + R'.$$

By the induction hypothesis,  $H'$  contains  $t$  disjoint subsets  $A_1, \dots, A_t$  of vertices so that for  $1 \leq i \leq t$ ,  $|A_i| = k+1$  and  $\delta(H'[A_i]) \geq 1$ . Choose  $U \subseteq N_H(v) \setminus \bigcup_{i=1}^t A_i$  so that  $|U| = k$  (note that, this is possible since  $d_H(v) \geq (k+1)(t+1)$ ). Clearly  $U \cup \{v\}$  is a new subset, disjoint from  $A_i$ s, of order  $k+1$  and minimum degree at least 1 in  $H$ . So we are done.

**Case 2.**  $A = \emptyset$ .

By the induction hypothesis,  $H$  contains  $t$  disjoint subsets  $A_1, \dots, A_t$  of vertices so that for  $1 \leq i \leq t$ ,  $|A_i| = k+1$  and  $\delta(H[A_i]) \geq 1$ .

Set  $\mathcal{A} = \bigcup_{i=1}^t A_i$  and  $H' = H \setminus \mathcal{A}$ . Since each vertex of  $H$  has degree less than  $(k+1)(t+1)$ , we have

$$e(H') \geq Rt + \binom{t}{2} + R' - t(k+1)((t+1)(k+1) - 1) + t(k+1)/2.$$

Since  $t \leq \lfloor \frac{kn - 2k^2}{(k+1)^2} \rfloor$ ,

$$Rt + \binom{t}{2} + t(k+1) + t(k+1)/2 - t(t+1)(k+1)^2 \geq 0.$$

So we have  $e(H') \geq R'$ . If there exists a subset  $A_{t+1}$  of  $V(H')$  so that  $|A_{t+1}| = k+1$  and  $\delta(H'[A_{t+1}]) \geq 1$ , then we are done. Otherwise, using Lemma 2.3 and Corollary 2.4, we may assume that  $k$  is even and  $H'$  contains a matching  $M$  so that  $|M| = e(H') \geq k(n-1)/2 + 1$ . But it is impossible, since  $|V(H')| = k(n-t-1) + 1$ . This completes the proof.  $\square$

Note that in the previous lemma we set  $n \geq 3k+3$  to guarantee  $\lfloor \frac{kn - 2k^2}{(k+1)^2} \rfloor \geq 1$ .

Let  $G$  be a graph so that  $G \longrightarrow (K_{1,k}, K_n)$ . In the following theorem we present a sufficient condition on  $G$  so that  $e(G) \geq \hat{r}^*(K_{1,k}, K_n)$ .

**Theorem 2.7.** *Let  $k \geq 2$  and  $n \geq 3k+3$ . If  $G \longrightarrow (K_{1,k}, K_n)$  and  $|G| = R + \ell$  so that  $R = r(K_{1,k}, K_n) = k(n-1) + 1$  and  $0 \leq \ell \leq \lfloor \frac{kn - 2k^2}{(k+1)^2} \rfloor$ , then  $e(G) \geq \hat{r}^*(K_{1,k}, K_n)$ .*

*Proof.* Suppose to the contrary that  $e(G) < \hat{r}^*(K_{1,k}, K_n)$ . So using Theorem 1.2, we have  $e(\overline{G}) \geq R\ell + \binom{\ell}{2} + R'$ , where

$$R' = \begin{cases} \binom{k}{2} + 1 & \text{if } k \text{ is odd,} \\ k(n-1)/2 + 1 & \text{if } k \text{ is even.} \end{cases}$$

Using Lemma 2.6,  $\overline{G}$  contains  $\ell + 1$  disjoint subsets  $A_1, \dots, A_{\ell+1}$  of vertices so that for  $1 \leq i \leq \ell + 1$ , we have  $|A_i| = k + 1$  and  $\delta(\overline{G}[A_i]) \geq 1$ . Now consider the following colouring on  $G$ . Partition  $V(G)$  into subsets  $X_1, \dots, X_{n-1}$  so that  $X_i = A_i$ , for  $1 \leq i \leq \ell + 1$  and  $|X_i| = k$ , for  $i = \ell + 2, \dots, n - 1$ . Colour every edge of  $G[X_i]$  red ( $1 \leq i \leq n - 1$ ) and all other edges of  $G$  blue. Then there is no red copy of  $K_{1,k}$  and no blue copy of  $K_n$ . Which is a contradiction with the assumption that  $G \rightarrow (K_{1,k}, K_n)$ .  $\square$

### 3 The proof of Theorem 1.4

*Proof.* Since  $\hat{r}(K_{1,k}, K_n) \leq \hat{r}^*(K_{1,k}, K_n)$ , we shall prove just the lower bound for the claimed size Ramsey number. Let  $k \geq 2$  and  $n \geq k^3 + 2k^2 + 2k$ . Also let  $G$  be a graph so that  $G \rightarrow (K_{1,k}, K_n)$ . Without loss of generality we may assume that  $G$  is edge minimal. Let  $|G| = R + \ell = k(n-1) + 1 + \ell$ , where  $\ell \geq 0$ . We will show that  $e(G) \geq \hat{r}^*(K_{1,k}, K_n)$ . If  $\ell \leq \lfloor \frac{kn - 2k^2}{(k+1)^2} \rfloor$ , then using Theorem 2.7, we are done. So we may

assume that  $\ell > \lfloor \frac{kn - 2k^2}{(k+1)^2} \rfloor$ .

Set  $f(k, n) = \lfloor \frac{kn - 2k^2}{(k+1)^2} \rfloor$  and for  $j = 1, \dots, n - 3$  set  $m_j = \max\{0, f(k, n - j)\}$ . Let  $T_0 = V(G)$ . Clearly  $G[T_0] \rightarrow (K_{1,k}, K_n)$  and  $|T_0| = k(n-1) + 1 + \ell_0$ , where  $\ell_0 = \ell$ . Repeat the following process as long as possible.

#### Step 1

Using Lemma 2.2, for  $S = \emptyset$  there exists  $T_1 \subset T_0$  such that  $\Delta(G[T_0 \setminus T_1]) < k$  and each vertex  $x \in T_1$  sends at least  $k$  edges to  $B_1 = T_0 \setminus T_1$ . Let  $T_1$  be such a set with the minimum number of vertices. Note that  $G[T_1] \rightarrow (K_{1,k}, K_{n-1})$ . To see this, colour the edges of  $G[T_1]$  arbitrarily and extend this colouring to  $G[T_0]$  by colouring the edges of  $B_1$  red and all so far uncoloured edges blue. Since  $G \rightarrow (K_{1,k}, K_n)$ , then  $G[T_1]$  contains a red copy of  $K_{1,k}$  or a blue copy of  $K_{n-1}$ . Therefore,  $|T_1| \geq r(K_{1,k}, K_{n-1}) = k(n-2) + 1$ . Let  $|T_1| = k(n-2) + 1 + \ell_1$ , where  $\ell_1 \geq 0$ . Clearly  $|B_1| = |T_0| - |T_1| = k + \ell_0 - \ell_1$ . Since each vertex of  $T_1$  sends at least  $k$  edges to  $B_1$ , so  $|B_1| \geq k$  which implies that  $\ell_1 \leq \ell_0$ . If  $\ell_1 \leq m_1$ , then stop. Otherwise go to Step 2.

#### Step i ( $2 \leq i \leq n - 3$ )

Since  $G[T_{i-1}] \rightarrow (K_{1,k}, K_{n-i+1})$  by Lemma 2.2, for  $S = \emptyset$  there exists  $T_i \subset T_{i-1}$  such that  $\Delta(G[T_{i-1} \setminus T_i]) < k$  and each vertex  $x \in T_i$  sends at least  $k$  edges to  $B_i = T_{i-1} \setminus T_i$ . Let  $T_i$  be such a set with the minimum number of vertices. Note that  $G[T_i] \rightarrow (K_{1,k}, K_{n-i})$ . To see this, colour the edges of  $G[T_i]$  arbitrarily and extend this colouring to  $G$  by

colouring the edges of  $G[B_1], \dots, G[B_i]$  red and all so far uncoloured edges blue. Since  $G \rightarrow (K_{1,k}, K_n)$ , then  $G[T_i]$  contains a red copy of  $K_{1,k}$  or a blue copy of  $K_{n-i}$ . Therefore,  $|T_i| \geq r(K_{1,k}, K_{n-i}) = k(n-i-1) + 1$ . Let  $|T_i| = k(n-i-1) + 1 + \ell_i$ , where  $\ell_i \geq 0$ . Clearly  $|B_i| = |T_{i-1}| - |T_i| = k + \ell_{i-1} - \ell_i$ . Since each vertex of  $T_i$  sends at least  $k$  edges to  $B_i$ , so  $|B_i| \geq k$  which implies that  $\ell_i \leq \ell_{i-1}$ . If either  $\ell_i \leq m_i$  or  $i = n-3$ , then stop. Otherwise go to Step  $i+1$ .

Now, assume that the above procedure terminates in step  $j$ . We have one of the following cases.

**Case 1.**  $j = 1$ .

As  $n \geq k^3 + 2k^2 + 2k$ , we have  $m_1 = f(k, n-1)$ . Since  $\ell_1 \leq f(k, n-1)$ , using Theorem 2.7, we have  $e(G[T_1]) \geq \hat{r}^*(K_{1,k}, K_{n-1})$ . We have two following subcases.

**Subcase 1.1**  $\ell_1 \geq \lceil k/2 \rceil$ .

In this case  $|T_1| \geq k(n-2) + 1 + \lceil \frac{k}{2} \rceil$ . Since each vertex  $x \in T_1$  sends at least  $k$  edges to  $B_1$ , we have

$$e(G) \geq e(G[T_1]) + k|T_1| \geq \hat{r}^*(K_{1,k}, K_{n-1}) + \frac{k^2(2n-3) + 2k}{2} \geq \hat{r}^*(K_{1,k}, K_n).$$

So we are done.

**Subcase 1.2**  $\ell_1 < \lceil k/2 \rceil$ .

Clearly  $|B_1| = k + \ell_0 - \ell_1 \geq k + \ell_0 - \lceil \frac{k}{2} \rceil + 1 = \ell_0 + \lfloor \frac{k}{2} \rfloor + 1$ . Using Remark 2.5 and the fact that  $\Delta(G[B_1]) < k$ , we conclude that each vertex of  $B_1$  sends at least  $n-k$  edges to  $T_1$ . Since  $n \geq k^3 + 2k^2 + 2k$ , we have

$$\begin{aligned} e(G) &\geq e(G[T_1]) + |B_1|(n-k) \geq \hat{r}^*(K_{1,k}, K_{n-1}) + (\ell_0 + \lfloor \frac{k}{2} \rfloor + 1)(n-k) \\ &\geq \hat{r}^*(K_{1,k}, K_{n-1}) + (\lfloor \frac{kn-2k^2}{(k+1)^2} \rfloor + \lfloor \frac{k}{2} \rfloor + 2)(n-k) \\ &\geq \hat{r}^*(K_{1,k}, K_{n-1}) + (\frac{kn-2k^2}{(k+1)^2} + \lfloor \frac{k}{2} \rfloor + 1)(n-k) \\ &\geq \hat{r}^*(K_{1,k}, K_{n-1}) + \frac{k^2(2n-3) + k}{2} \geq \hat{r}^*(K_{1,k}, K_n). \end{aligned}$$

**Case 2.**  $2 \leq j \leq n-3k-3$ .

In this case we have  $m_j = f(k, n-j)$ . Since  $\ell_j \leq m_j$ , using Theorem 2.7, we have

$e(G[T_j]) \geq \hat{r}^*(K_{1,k}, K_{n-j})$ . Therefore,

$$\begin{aligned}
e(G) &\geq e(G[T_j]) + |T_j|kj + \sum_{i=2}^j |B_i|k(i-1) \\
&\geq \hat{r}^*(K_{1,k}, K_{n-j}) + (k(n-j-1) + 1 + \ell_j)kj + \sum_{i=2}^j (k + \ell_{i-1} - \ell_i)(i-1)k \\
&= \hat{r}^*(K_{1,k}, K_{n-j}) + k^2j(n-j-1) + kj + kj\ell_j + k^2 \sum_{i=2}^j (i-1) + k \sum_{i=2}^j (\ell_{i-1} - \ell_i)(i-1) \\
&= \hat{r}^*(K_{1,k}, K_{n-j}) + k^2j(n-j-1) + kj + \frac{k^2j(j-1)}{2} + k \sum_{i=1}^j \ell_i.
\end{aligned}$$

If  $k$  is even, using Theorem 1.2, we have  $\hat{r}^*(K_{1,k}, K_{n-j}) = k^2(n-j-1)^2/2$ . So

$$\begin{aligned}
e(G) &\geq \frac{k^2(n-j-1)^2}{2} + k^2j(n-j-1) + kj + \frac{k^2j(j-1)}{2} + k \sum_{i=1}^j \ell_i \\
&= \frac{k^2(n-1)^2}{2} - \frac{k^2j}{2} + kj + k \sum_{i=1}^j \ell_i.
\end{aligned}$$

Note that if  $k$  is even, then  $\hat{r}^*(K_{1,k}, K_n) = k^2(n-1)^2/2$ . So it suffices to show that

$$k \sum_{i=1}^j \ell_i + kj - \frac{k^2j}{2} \geq 0.$$

Since for  $1 \leq i \leq j-1$ , we have  $\ell_i \geq \lfloor \frac{k(n-i) - 2k^2}{(k+1)^2} \rfloor + 1 \geq \frac{k(n-i) - 2k^2}{(k+1)^2}$ , therefore

$$\begin{aligned}
k \sum_{i=1}^j \ell_i + kj - \frac{k^2j}{2} &\geq k \sum_{i=1}^{j-1} \frac{k(n-i) - 2k^2}{(k+1)^2} + kj - \frac{k^2j}{2} \\
&= k \left( \frac{kn(j-1) - k \frac{j(j-1)}{2} - 2k^2(j-1)}{(k+1)^2} \right) - \frac{(k^2 - 2k)j}{2} \\
&= \frac{k(j-1)(2kn - kj - 4k^2 - (k-2)(k+1)^2 \frac{j}{j-1})}{2(k+1)^2} \geq 0.
\end{aligned}$$

The last inequality is true since  $n \geq k^3 + 2k^2 + 2k$  and  $2 \leq j \leq n - 3k - 3$  imply

$$2kn - kj - 4k^2 - (k-2)(k+1)^2 \frac{j}{j-1} \geq 0.$$

So we are done.

If  $k$  is odd, then  $\hat{r}^*(K_{1,k}, K_n) = \binom{k(n-1)+1}{2} - \binom{k}{2}$ . To verify that  $e(G) \geq \hat{r}^*(K_{1,k}, K_n)$ , it suffices to show that

$$k \sum_{i=1}^j \ell_i + \frac{kj - k^2j}{2} \geq 0.$$



The above inequality follows from a similar argument that used for the case  $k$  is even.

**Case 3.**  $n - 3k - 3 < j \leq n - 4$ .

In this case  $m_j = 0$ . Note that  $G[T_j] \longrightarrow (K_{1,k}, K_{n-j})$  and  $|T_j| = r(K_{1,k}, K_{n-j}) + \ell_j$ . Since  $0 \leq \ell_j \leq m_j$ , we have  $|T_j| = r(K_{1,k}, K_{n-j})$ . Now, by the definition of  $\hat{r}^*(K_{1,k}, K_{n-j})$ , we have  $e(G[T_j]) \geq \hat{r}^*(K_{1,k}, K_{n-j})$ . By an argument similar to Case 2, we have

$$\begin{aligned} e(G) &\geq e(G[T_j]) + |T_j|kj + \sum_{i=2}^j |B_i|k(i-1) \\ &\geq \hat{r}^*(K_{1,k}, K_{n-j}) + k^2j(n-j-1) + kj + \frac{k^2j(j-1)}{2} + k \sum_{i=1}^{j-1} \ell_i. \end{aligned}$$

Our aim is to show that  $e(G) \geq \hat{r}^*(K_{1,k}, K_n)$ . If  $k$  is even, similar to Case 2, it suffices to show that

$$k \sum_{i=1}^{j-1} \ell_i + kj - \frac{k^2j}{2} \geq 0.$$

Note that for  $1 \leq i \leq j-1$ ,  $\ell_i > m_i$  and

$$m_i = \begin{cases} f(k, n-i) & 1 \leq i \leq n-3k-3, \\ 0 & n-3k-2 \leq i \leq j-1. \end{cases}$$

Since for  $1 \leq i \leq n-3k-3$ , we have  $\ell_i \geq \lfloor \frac{k(n-i)-2k^2}{(k+1)^2} \rfloor + 1 \geq \frac{k(n-i)-2k^2}{(k+1)^2}$ , therefore

$$\begin{aligned} k \sum_{i=1}^{j-1} \ell_i + kj - \frac{k^2j}{2} &\geq k \sum_{i=1}^{n-3k-3} \frac{k(n-i)-2k^2}{(k+1)^2} + k \sum_{i=n-3k-2}^{j-1} 1 - \frac{(k^2-2k)j}{2} \\ &> k \sum_{i=1}^{n-3k-3} \frac{k(n-i)-2k^2}{(k+1)^2} - \frac{(k^2-2k)j}{2} \\ &= k \left( \frac{k(n-3k-3)(n-\frac{n-3k-2}{2}-2k)}{(k+1)^2} \right) - \frac{(k^2-2k)j}{2} \\ &= k \left( \frac{k(n-3k-3)(n-k+2) - (k+1)^2(k-2)j}{2(k+1)^2} \right) \geq 0. \end{aligned}$$

The last inequality is true, since  $n \geq k^3 + 2k^2 + 2k$  and  $n-3k-2 \leq j \leq n-4$  imply

$$k(n-3k-3)(n-k+2) - (k+1)^2(k-2)j \geq 0.$$

So when  $k$  is even, we are done.

Similarly, when  $k$  is odd, it can be shown that  $e(G) \geq \hat{r}^*(K_{1,k}, K_n)$ .

**Case 4.**  $j = n - 3$ .

In this case for every  $1 \leq i \leq n-4$ , we have  $\ell_i > m_i$ . Note that,  $G[T_{n-3}] \longrightarrow (K_{1,k}, K_3)$  and

$$|T_{n-3}| = r(K_{1,k}, K_3) + \ell_{n-3} = 2k + 1 + \ell_{n-3}.$$

Using Theorem 2.1, we have  $e(G[T_{n-3}]) \geq k^2$ . By an argument similar to Case 2, we have

$$\begin{aligned} e(G) &\geq e(G[T_{n-3}]) + |T_{n-3}|k(n-3) + \sum_{i=2}^{n-3} |B_i|k(i-1) \\ &\geq k^2 + (2k+1+\ell_{n-3})k(n-3) + \sum_{i=2}^{n-3} (k+\ell_{i-1}-\ell_i)(i-1)k \\ &= k^2 + 2k^2(n-3) + k(n-3) + \frac{k^2(n-3)(n-4)}{2} + k \sum_{i=1}^{n-3} \ell_i. \end{aligned}$$

Again, we are going to show that  $e(G) \geq \hat{r}^*(K_{1,k}, K_n)$ . When  $k$  is even, we have  $\hat{r}^*(K_{1,k}, K_n) = k^2(n-1)^2/2$ . It suffices to show that

$$k \sum_{i=1}^{n-3} \ell_i + k^2 + 2k^2(n-3) + k(n-3) \geq \frac{k^2}{2}(5n-11).$$

This inequality is certainly true if

$$2k \sum_{i=1}^{n-3} \ell_i + k^2 + 2kn \geq k^2n + 6k.$$

By an argument similar to Case 3, we have

$$\begin{aligned} 2k \sum_{i=1}^{n-3} \ell_i + k^2 + 2kn &\geq 2k \sum_{i=1}^{n-3k-3} \frac{k(n-i)-2k^2}{(k+1)^2} + 2k \sum_{i=n-3k-2}^{n-4} 1 + k^2 + 2kn \\ &= \frac{k^2(n-3k-3)(n-k+2)}{(k+1)^2} + 2k(3k-1) + k^2 + 2kn \\ &= \frac{k^2(n-3k-3)(n-k+2) + (7k^2-2k+2kn)(k+1)^2}{(k+1)^2} \\ &\geq k^2n + 6k. \end{aligned}$$

The last inequality holds, since  $n \geq k^3 + 2k^2 + 2k$ .

If  $k$  is odd, then  $\hat{r}^*(K_{1,k}, K_n) = \binom{k(n-1)+1}{2} - \binom{k}{2}$ . In this case, it suffices to show that

$$2k \sum_{i=1}^{n-3} \ell_i + 2k^2 + kn \geq k^2n + 6k.$$

Again, since  $n \geq k^3 + 2k^2 + 2k$ , the above inequality holds. So we are done and the proof is completed. □

## References

- [1] P. Erdős, Problems and results in graph theory, The theory and applications of graphs (G. Chartrand, ed.), *John Wiley*, New York, 1981, pp. 331–341.
- [2] P. Erdős, R. Faudree, C. Rousseau and R. Schelp, The size Ramsey number, *Period. Math. Hungar.* 9 (1978), 145–161.
- [3] R.J. Faudree and J. Sheehan, Size Ramsey numbers for small-order graphs, *J. Graph Theory* 7 (1983), 53–55.
- [4] R.J. Faudree and J. Sheehan, Size Ramsey numbers involving stars, *Discrete Math.* 46 (1983), 151–157.
- [5] O. Pikhurko, Size Ramsey numbers of stars versus 3-chromatic graphs, *Combinatorica* 21 (2001), 403–412.
- [6] O. Pikhurko, Size Ramsey numbers of stars versus 4-chromatic graphs, *J. Graph Theory* 42 (2003), 220–233.